Abramov, V. M. (2002). Asymptotic analysis of the GI/M/1/n loss system as n increases to infinity.

Originally published in *Annals of Operations Research, 112*(1–4), 35–41.
Available from: http://dx.doi.org/10.1023/a:1020920819656

Asymptotic analysis of the GI/M/1/n loss system as n increases to infinity

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Abstract. This paper provides the asymptotic analysis of the loss probability in the $GI/M/1/n$ queueing system as n increases to infinity. The approach of this paper is alternative to that of the recent papers of Choi and Kim [8] and Choi et al [9] and based on application of modern Tauberian theorems with remainder. This enables us to simplify the proofs of the results on asymptotic behavior of the loss probability of the abovementioned paper of Choi and Kim [9] as well as to obtain some new results.

Keywords: Loss system, $GI/M/1/n$ queue, asymptotic analysis, Tauberian theorems with remainder

1. Introduction

Consider $GI/M/1/n$ queueing system denoting by $A(x)$ the probability distribution function of interarrival time and by λ the reciprocal of the expected interarrival time, $\alpha(s)=\int_0^\infty \mathrm{e}^{-sx}\mathrm{d}A(x)$. The parameter of the service time distribution will be denoted μ, and load of the system is $\rho=\lambda/\mu$. The size of buffer n includes the position for server. Denote also $\rho_m=\mu^m\int_0^\infty x^m\mathrm{d}A(x)$, $m=1,2,...$, $(\rho_1=\rho^{-1})$.

The explicit representation for the loss probability in terms of generating function was obtained by Miyazawa [12]. Namely, he showed that whenever the value of load ρ, the loss probability p_n always exists and has the representation

$$p_n=\frac{1}{\sum_{j=0}^n \pi_j}, \tag{1.1}$$

where the generating function $\Pi(z)$ of π_j, $j=0,1,...$, is the following

$$\Pi(z)=\sum_{j=0}^\infty \pi_j z^j=\frac{(1-z)\alpha(\mu-\mu z)}{\alpha(\mu-\mu z)-z},\quad |z|<\sigma, \tag{1.2}$$

σ is the minimum nonnegative solution of the functional equation $z=\alpha(\mu-\mu z)$. This solution is the following. It belongs to the open interval (0,1) if $\lambda<\mu$, and it is equal to 1 otherwise.

In the recent papers Choi and Kim [8] and Choi et al [9] study the questions related to the asymptotic behavior of the sequence $\{\pi_j\}$ as

$j \to \infty$. Namely, they study asymptotic behavior of the loss probability p_n, $n \to \infty$, as well as obtain the convergence rate of the stationary distributions of the $GI/M/1/n$ queueing system to those of the $GI/M/1$ queueing system as $n \to \infty$. The analysis of [8] and [9] is based on the theory of analytic functions.

The approach of this paper is based on Tauberian theorems with remainder permitting us to simplify the proof of the results of the mentioned paper of Choi et al [9] as well as to obtain some new results on asymptotic behavior of the loss probability.

For the asymptotic behavior of the loss probability in $M/GI/1/n$ queue see Abramov [1], [2], Asmussen [6], Takagi [16], Tomko [17], Willmot [18] etc. For the asymptotic analysis of more general than $M/GI/1/n$ queueing systems see Abramov [3], Baiocchi [7] etc.

Study of the loss probability and its asymptotic analysis is motivated by growing development of communication systems. The results of our study can be applied to the problems of flow control, performance evaluation, redundancy. For application of the loss probability to such kind of problems see Ait-Hellal et al [4], Altman and Jean-Marie [5], Cidon et al [10], Gurewitz et al [11].

2. Auxiliary results. Tauberian theorems

In this section we represent the asymptotic results of Takacs [15, p.22-23] (see Lemma 2.1 below), and Tauberian theorems of Postnikov [13, Section 25], (see Lemmas 2.2 and 2.3 below).

Let Q_j, $j = 0, 1, ...$, be a sequence of real numbers satisfying the recurrent relation

$$Q_n = \sum_{j=0}^{n} r_j Q_{n-j+1}, \tag{2.1}$$

where r_j, $j = 0, 1, ...$, are nonnegative numbers, $r_0 > 0$, $r_0 + r_1 + ... = 1$, and $Q_0 > 0$ is an arbitrary real number.

Denote $r(z) = \sum_{j=0}^{\infty} r_j z^j$, $|z| \leq 1$, $\gamma_m = r^{(m)}(1-0) = \lim_{z \uparrow 1} r^{(m)}(z)$, where $r^{(m)}(z)$ is the mth derivative of $r(z)$. Then for $Q(z) = \sum_{j=0}^{\infty} Q_j z^j$, the generating function of Q_j, $j = 0, 1, ...$, we have the following representation

$$Q(z) = \frac{Q_0 r(z)}{r(z) - z}. \tag{2.2}$$

The statements below are known theorems on asymptotic behavior of the sequence $\{Q_j\}$ as $j \to \infty$. Lemma 2.1 below joins two results by

Takacs [15]: Theorem 5 on p. 22 and relation (35) on p. 23.

Lemma 2.1 (Takacs [15]). *If* $\gamma_1 < 1$ *then*

$$\lim_{n\to\infty} Q_n = \frac{Q_0}{1-\gamma_1}.$$

If $\gamma_1 = 1$ *and* $\gamma_2 < \infty$ *then*

$$\lim_{n\to\infty} \frac{Q_n}{n} = \frac{2Q_0}{\gamma_2}.$$

If $\gamma_1 > 1$ *then*

$$\lim_{n\to\infty} \Big(Q_n - \frac{Q_0}{\delta^n[1-r'(\delta)]}\Big) = \frac{Q_0}{1-\gamma_1},$$

where δ *is the least (absolute) root of equation* $z = r(z)$.

Lemma 2.2 (Postnikov [13]). *Let* $\gamma_1 = 1$ *and* $\gamma_3 < \infty$. *Then as* $n \to \infty$

$$Q_n = \frac{2Q_0}{\gamma_2} n + O(\log n).$$

Lemma 2.3 (Postnikov [13]). *Let* $\gamma_1 = 1$, $\gamma_2 < \infty$ *and* $r_0 + r_1 < 1$. *Then as* $n \to \infty$

$$Q_{n+1} - Q_n = \frac{2Q_0}{\gamma_2} + o(1).$$

3. The main results on asymptotic behavior of the loss probability

Let us study (1.1) and (1.2) more carefully. Represent (1.2) as the difference of two terms

$$\Pi(z) = \frac{(1-z)\alpha(\mu-\mu z)}{\alpha(\mu-\mu z)-z} = \frac{\alpha(\mu-\mu z)}{\alpha(\mu-\mu z)-z} - z\frac{\alpha(\mu-\mu z)}{\alpha(\mu-\mu z)-z}$$

$$= \widetilde{\Pi}(z) - z\widetilde{\Pi}(z), \tag{3.1}$$

where

$$\widetilde{\Pi}(z) = \sum_{j=0}^{\infty} \widetilde{\pi}_j z^j = \frac{\alpha(\mu-\mu z)}{\alpha(\mu-\mu z)-z}. \tag{3.2}$$

Note also that

$$\pi_0 = \widetilde{\pi}_0 = 1,$$

$$\pi_{j+1} = \widetilde{\pi}_{j+1} - \widetilde{\pi}_j, \quad j \geq 0. \tag{3.3}$$

Therefore,

$$\sum_{j=0}^{n} \pi_j = \widetilde{\pi}_n,$$

and

$$p_n = \frac{1}{\widetilde{\pi}_n}. \tag{3.4}$$

Now the application of Lemma 2.1 yields the following

Theorem 3.1. *In the case where $\rho < 1$ as $n \to \infty$ we have*

$$p_n = \frac{(1-\rho)[1+\mu\alpha'(\mu-\mu\sigma)]\sigma^n}{1-\rho-\rho[1+\mu\alpha'(\mu-\mu\sigma)]\sigma^n} + o(\sigma^{2n}). \tag{3.5}$$

In the case where $\rho_2 < \infty$ and $\rho = 1$ we have

$$\lim_{n\to\infty} np_n = \frac{\rho_2}{2}. \tag{3.6}$$

In the case where $\rho > 1$ we have

$$\lim_{n\to\infty} p_n = \frac{\rho - 1}{\rho}. \tag{3.7}$$

Proof. Indeed, it follows from (3.1), (3.2) and (3.3) that $\widetilde{\pi}_0 = 1$, and

$$\widetilde{\pi}_k = \sum_{i=0}^{k} \frac{(-\mu)^i}{i!} \alpha^{(i)}(\mu) \widetilde{\pi}_{k-i+1}, \tag{3.8}$$

where $\alpha^{(i)}(\mu)$ denotes the ith derivative of $\alpha(\mu)$. Note also that $\alpha(\mu) > 0$, the terms $(-\mu)^i \alpha^{(i)}(\mu)/i!$ are nonnegative for all $i \geq 1$, and

$$\sum_{i=0}^{\infty} \frac{(-\mu)^i}{i!} \alpha^{(i)}(\mu) = \sum_{i=0}^{\infty} \int_0^{\infty} \mathrm{e}^{-\mu x} \frac{(\mu x)^i}{i!} \mathrm{d}A(x)$$
$$= \int_0^{\infty} \sum_{i=0}^{\infty} \mathrm{e}^{-\mu x} \frac{(\mu x)^i}{i!} \mathrm{d}A(x) = 1. \tag{3.9}$$

Therefore one can apply Lemma 2.1. Then in the case of $\rho < 1$ one can write

$$\lim_{n\to\infty} \Big(\widetilde{\pi}_n - \frac{1}{\sigma^n[1+\mu\alpha'(\mu-\mu\sigma)]}\Big) = \frac{\rho}{\rho-1}, \tag{3.10}$$

and for large n relation (3.10) can be rewritten in the form of the estimation

$$\widetilde{\pi}_n = \Big[\frac{1}{\sigma^n[1+\mu\alpha'(\mu-\mu\sigma)]} + \frac{\rho}{\rho-1}\Big][1+o(\sigma^n)]. \tag{3.11}$$

In turn, from (3.11) for large n we obtain

$$p_n = \frac{1}{\widetilde{\pi}_n} = \frac{(1-\rho)[1+\mu\alpha'(\mu-\mu\sigma)]\sigma^n}{1-\rho-\rho[1+\mu\alpha'(\mu-\mu\sigma)]\sigma^n}[1+o(\sigma^n)]$$

$$= \frac{(1-\rho)[1+\mu\alpha'(\mu-\mu\sigma)]\sigma^n}{1-\rho-\rho[1+\mu\alpha'(\mu-\mu\sigma)]\sigma^n} + o(\sigma^{2n}).$$

Thus (3.5) is proved. The limiting relations (3.6) and (3.7) follow immediately by application of Lemma 2.1. Theorem 3.1 is proved.

The following two theorems improve limiting relation (3.6). From Lemma 2.2 we have the following

Theorem 3.2. *Assume that $\rho = 1$ and $\rho_3 < \infty$. Then as $n \to \infty$*

$$p_n = \frac{\rho_2}{2n} + O\Big(\frac{\log n}{n^2}\Big). \tag{3.12}$$

Proof. The result follows immediately by application of Lemma 2.2.

Subsequently, from Lemma 2.3 we have

Theorem 3.3. *Assume that $\rho = 1$ and $\rho_2 < \infty$. Then as $n \to \infty$*

$$\frac{1}{p_{n+1}} - \frac{1}{p_n} = \frac{2}{\rho_2} + o(1). \tag{3.13}$$

Proof. The theorem will be proved if we show that for all $\mu > 0$

$$\alpha(\mu) - \mu\alpha'(\mu) < 1. \tag{3.14}$$

Taking into account (3.9) and the fact that $(-\mu)^i\alpha^{(i)}(\mu)/i! \geq 0$ for all $i \geq 0$, one can write

$$\alpha(\mu) - \mu\alpha'(\mu) \leq 1. \tag{3.15}$$

Thus, we have to show that for some $\mu_0 > 0$ the equality

$$\alpha(\mu_0) - \mu_0\alpha'(\mu_0) = 1 \tag{3.16}$$

is not a case. Indeed, since $\alpha(\mu) - \mu\alpha'(\mu)$ is an analytic function then, according to the theorem on maximum absolute value of analytic function, the equality $\alpha(\mu) - \mu\alpha'(\mu) = 1$ is valid for all $\mu > 0$. This means that (3.16) is valid if and only if $\alpha^{(i)}(\mu) = 0$ for all $i \geq 2$ and for all $\mu > 0$, and therefore $\alpha(\mu)$ is a linear function, i.e. $\alpha(\mu) = c_0 + c_1\mu$, where c_0 and c_1 are some constants. However, since $|\alpha(\mu)| \leq 1$ we obtain $c_0 = 1$, $c_1 = 0$. This is a trivial case where the probability distribution function $A(x)$ is concentrated in point 0. Therefore (3.16) is not a case, and hence (3.14) holds. Theorem 3.3 is proved.

We have also the following

Theorem 3.4. *Let $\rho = 1 - \epsilon$, where $\epsilon > 0$, and $\epsilon n \to C > 0$ as $n \to \infty$ and $\epsilon \to 0$. Assume that $\rho_3 = \rho_3(n)$ is a bounded function and there exists $\widetilde{\rho}_2 = \lim_{n\to\infty} \rho_2(n)$. Then,*

$$p_n = \frac{\epsilon e^{-2C/\widetilde{\rho}_2}}{1 - e^{-2C/\widetilde{\rho}_2}}[1 + o(1)]. \tag{3.17}$$

Proof. It was shown in Subhankulov [14, p. 326] that if $\rho^{-1} = 1+\epsilon$, $\epsilon > 0$ and $\epsilon \to 0$, $\rho_3(n)$ is a bounded function, and there exists $\widetilde{\rho}_2 = \lim_{n\to\infty} \rho_2(n)$ then

$$\sigma = 1 - \frac{2\epsilon}{\widetilde{\rho}_2} + O(\epsilon^2), \tag{3.18}$$

where $\sigma = \sigma(n)$ is the minimum root of the functional equation $z - \alpha(\mu - \mu z) = 0$, $|z| \leq 1$, and where the parameter μ and the function $\alpha(z)$, both or one of them, are assumed to depend on n. Therefore, (3.18) is also valid under the assumptions of the theorem. Then after some algebra one can obtain

$$[1 + \mu\alpha'(\mu - \mu\sigma)]\sigma^n = \epsilon \mathrm{e}^{-2C/\widetilde{\rho}_2}[1 + o(1)],$$

and the result easily follows from estimation (3.11).

Theorem 3.5. *Let $\rho = 1 - \epsilon$, where $\epsilon > 0$, and $\epsilon n \to 0$ as $n \to \infty$ and $\epsilon \to 0$. Assume that $\rho_3 = \rho_3(n)$ is a bounded function and there exists $\widetilde{\rho}_2 = \lim_{n\to\infty} \rho_2(n)$. Then*

$$p_n = \frac{\widetilde{\rho}_2}{2n} + o\Big(\frac{1}{n}\Big). \tag{3.19}$$

Proof. The proof follows by expanding of the main term of asymptotic relation (3.17) for small C.

4. Discussion

We obtained a number of asymptotic results related to the loss probability for the $GI/M/1/n$ queueing system by using Tauberian theorems with remainder. Asymptotic relations (3.6) and (3.7) of Theorem 3.1 are the same as correspondent asymptotic relations of Theorem 3 of [9]. Asymptotic relation (3.5) of Theorem 3.1 improves correspondent asymptotic relation of Theorem 3 of [9], however it can be deduced from Theorem 3.1 of [8] and the second equation on p. 1016 of [8]. Under additional condition $\rho_3 < \infty$ the statement (3.12) of Theorem 3.2 is new. It improves the result of [9] under $\rho = 1$: the remainder obtained in Theorem 3.2 is $O(\log n/n^2)$ whereas under condition $\rho_2 < \infty$ the remainder obtained in Theorem 3 of [9] is $o(n^{-1})$. Asymptotic relation (3.13) of Theorem 3.3 coincides with intermediate asymptotic relation on p. 441 of [9]. Theorems 3.4 and 3.5 are new. They provide asymptotic results where the load ρ is close to 1.

Acknowledgement

The author thanks the anonymous referees for a number of valuable comments.

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